$$\times\sqrt{[a^2c^2+b^2(x-a)^2]}-a^3(2c^2-b^2)\sqrt{[b^2+c^2]}+dc^3(d^2-3a^2)$$

$$+3a^2bc^3\log[c(b+d)]-3a^3bc^2\log a-3a^2bc^2(x-a)\log[b(x-a)$$

$$+\sqrt{[a^2c^2+b^2(x-a)^2]}]-3a^3bc^2\log[\sqrt{(b^2+c^2)}-b]-3a^2bc^2(u-a)\log[b(u-a)$$

$$+\sqrt{[a^2c^2+b^2(u-a)^2]}]\}dxdu$$

$$=\frac{1}{72a^5b^2}\int_0^a\Big[24x[2a^2x^2-3b^3(x-a)^2]\sqrt{[a^2x^2+b^2(x-a)^2]}-16a^3bx^3$$

$$-16a^3b(x-a)^3+(x-a)^4(72b^2d-12a^2d-12d^3-23a^2b)+x^4(12d^3-36a^2d$$

$$+23a^2b)+12a^3x(3b^3-2x^2)\sqrt{[b^2+x^2]}+12a^3(x-a)[2(x-a)^2-3b^2]$$

$$\times\sqrt{[b^2+(x-a)^2]}+60a^2b(x-a)^4\log[(b+d)(x-a)]$$

$$+\frac{24b^4(x-a)^4}{a}\log[(a+d)(x-a)]-96a^2bx^3(x-a)\log\{b(x-a)$$

$$+\sqrt{[a^2x^2+b^2(x-a)^2]}\}-48a^3bx^3\log[\sqrt{(b^2+c^2)}-b]-\frac{24b^4(x-a)^4}{a}$$

$$\times\log\{ax+\sqrt{[a^2x^2+b^2(x-a)^2]}\}+48a^3b(x-a)^3\log\{\sqrt{[b^2+(x-a)^2]}-b\}$$

$$+12a^3b^4\log[x+\sqrt{(b^2+x^2)}]-12a^3b^4\log\{x-a+\sqrt{[b^2+(x-a)^2]}\}$$

$$+36a^2bx^4\log[x(b+d)]-48a^3bx^3\log a+48a^3b(x-a)^3\log a\Big]dx$$

$$=\tfrac{1}{15}\Big[2d\Big(2-\frac{a^2}{b^2}-\frac{b^2}{a^2}\Big)-\frac{a^3+b^3}{d^2}+\frac{2a^3}{b^2}+\frac{2b^3}{a^2}+\Big(\frac{4b^2}{a}-\frac{a^2b^2}{d^3}\Big)\log\Big(\frac{a+d}{b}\Big)$$

$$+\Big(\frac{4a^2}{b}-\frac{a^2b^2}{d^3}\Big)\log\Big(\frac{b+d}{a}\Big)\Big]=\frac{a}{30}[6+(16-\sqrt{2})\log(1+\sqrt{2})],\text{ if } a=b.$$

MISCELLANEOUS.

135. Proposed by LON C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

Find invariants of the second, third, and sixth degrees in the coefficients of a binary quartic.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4=u=0$, be the binary quartic.

Then $d^4u/dx^4=24a$, $d^4u/dx^3dy=24b$, $d^4u/dx^2dy^2=24c$, $d^4u/dxdy^3=24d$, $d^4u/dy^4=24e$.

Substitute in u, d^4u/dx^4 for y^4, d^4u/dy^4 for x^4, $-d^4u/dx^3dy$ for xy^3, $-d^4u/dxdy^3$ for x^3y, d^4u/dx^2dy^2 for x^2y^2, and we get $24ae-96bd+144c^2-96bd+24ae=48(ae-4bd+3c^2)$, the invariant of second order.

$d^2u/dx^2=12(ax^2+2bxy+cy^2)$.

$d^2u/dy^2=12(cx^2+2dxy+ey^2)$.

$d^2u/dxdy=12(bx^2+2cxy+dy^2)$.

$\therefore$ The Hessian of this quartic is

$$(ax^2+2bxy+cy^2)(cx^2+2dxy+ey^2)-(bx^2+2cxy+dy^2)^2=0, \text{ or } (ac-b^2)x^4$$
$$+2(ad-bc)x^3y+(ae+2bd-3c^2)x^2y^2+2(be-cd)xy^3+(ce-d^2)y^4=0.$$

Write this in the form $Ax^4+4Bx^3y+6Cx^2y^2+4Dxy^3+Ey^4=0$.

Then $24(aE-4bD+6cC-4dB+Ae)=72(ace+2bcd-ad^2-eb^2-c^3)$, is the invariant of the third order.

$\therefore S=ae-4bd+3c^2$, is the invariant of the second order.

$T=ace+2bcd-ad^2-eb^2-c^3$ is the invariant of the third order.

These are the only two ordinary invariants. The invariant of the sixth order being the discriminant which we determine as below.

If we had used the more general form $Ax^4+By^4+Cz^4$, where $x+y+z=0$, then we would have had $a=A+C$, $e=B+C$, $b=C=e=d$.

$\therefore S=BC+CA+AB$, $T=ABC$.

Equating to zero the two differential equations Ax^3-Cz^3, and By^3-Cz^3, we get $Ax^3=By^3=Cz^3$, or $ABCx^3/BC=ABCy^3/AC=ABCz^3/AB$.

$\therefore x^3 : y^3 : z^3=BC : AC : AB$, this in $x+y+z=0$ gives $(BC)^{\frac{1}{3}}+(AC)^{\frac{1}{3}}+(AB)^3=0$.

$\therefore (BC+AC+AB)^3-27A^2B^2C^2=0$, or $S^3-27T^2=0$.

$\therefore (ae-4bd+3c^2)^3-27(ace+2bcd-ad^2-eb^2-c^3)^2=$the sixth order.

Also solved by *G. W. GREENWOOD*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

169. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

At what rate per cent. must a note be discounted at the end of every quarter of a year in order to produce a discount equivalent to 10% interest for the year?